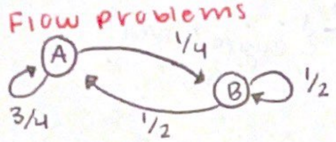
Transition Matrices

Flow problems



$$T = \begin{bmatrix} x_A \\ x_B \end{bmatrix} = \begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix}$$

Things to know

- if all columns add to 1, the system is conservative
- the inverse of T (T^{-1}) does not have the same network w/ flipped arrows
  - ↳ if T has no $\lambda = 0$, T is invertible so matrix C exists where $\vec{s}[n-1] = C\vec{s}[n]$
- Steady state solution
  - ↳ $x_f = \lim_{n \to \infty} T^n \vec{x}[1]$
  - ↳ should satisfy $T\vec{x}_f = \vec{x}_f$
  - ↳ if eigvals < 0, $\lim = 0$
  - ↳ if eigvals $= 0$, lim depends on initial state
  - ↳ if eigvals > 0, $\lim \to \infty$

proofs

Strategy

~~Example~~

① write out any mathematical definitions for what you ~~know~~ know and what you need to show

② try simple examples to find patterns

③ manipulate the definitions to get from what you know to what to show

④ proof by contradiction

↳ assume the opposite of what you need to show, demonstrate that it contradicts what you know!

Example #1

consider square matrix A. Prove that if A has a non trivial nullspace, then matrix A is not invertible

$\underbrace{\text{non trivial}}_{\text{the nullspace contains more than just } \vec{0}}$

know: A is square

A has nontrivial null space, aka

$A\vec{x} = \vec{0}$ for some $\vec{x} \neq \vec{0}$

show: A is NOT invertible

↳ $A^{-1}A = I$

prove by contradiction!

assume A is invertible...

$A^{-1}A = I$

$$A\vec{x} = \vec{0}$$
$$A^{-1}A\vec{x} = A^{-1}\vec{0}$$
$$I\vec{x} = A^{-1}\vec{0}$$
$$\vec{x} = \vec{0}$$

shows that if A is invertible, $\vec{x} = \vec{0}$ + $A\vec{x} = \vec{0}$, so by contradiction:

if A is NOT invertible it can't have $\vec{x} = \vec{0}$ so A is not invertible!

S is a subspace if:

① S contains $\vec{0}$ for some $A\vec{x} = \vec{0}$

② $\vec{x}$ in S $= c\vec{x}$ in S (scalar mult)

③ $\vec{a}$ in S, $\vec{b}$ in S, $\vec{a} + \vec{b}$ in S (vector addition)

Example #2

Given matrix $A^{n \times n}$ with unequal eigenvalues $\lambda_1 \neq \lambda_2$ and corresponding eigenspaces V_1, V_2, the bases of V_1 + V_2 are linearly independent from each other.

know: $\lambda_1 \neq \lambda_2$

show: V_1 + V_2 bases are lin. indep.

prove by contradiction!

assume the bases are linearly dependent on each other, so $\vec{v}_1 = a v_2$, $a \in \mathbb{R}$

*exclude $\vec{0}$ since that is in all vector spaces by def.

$$M\vec{v}_1 = M\vec{v}_1$$
$$M\vec{v}_1 = M(a\vec{v}_2)$$
$$\lambda_1 \vec{v}_1 = a(M\vec{v}_2)$$
$$\lambda_1 \vec{v}_1 = a(\lambda_2 \vec{v}_2)$$
$$\lambda_1 (a\vec{v}_2) = a(\lambda_2 \vec{v}_2)$$
$$\vec{0} = (\lambda_1 a\vec{v}_2) - (\lambda_2 a v_2)$$
$$\vec{0} = a v_2 (\lambda_1 - \lambda_2)$$

↑ must be $\lambda_1 = \lambda_2$ if bases are lin. dep.

∴ proven!

Basis + Linear independence

$$V = \left\{ \vec{x} \in \mathbb{R}^4 \;\middle|\; \vec{x} = \begin{bmatrix} \alpha \\ \beta \\ a+\beta \\ a-\beta \end{bmatrix}, \text{ where } \alpha, \beta \in \mathbb{R} \right\}$$

$$\vec{x} = \alpha \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

dimension = 2

cuz

$$\text{basis} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

↖ (lin indep set $\vec{v}$)

Proof!

know: $a_1 v_1 + a_2 v_2 \cdots + a_n v_n = 0 \to a_1, a_2, a_3, = 0$

prove: $\beta v_1 + \beta_2 (v_2 - v_1) + \beta_3 (v_3 - v_2 - v_1) + \ldots = 0$

Proof by contradiction

suppose not lin indep

which means $\beta v_1 + \beta_2 (v_2 - v_1) \ldots + \ldots = 0$

$\beta \neq \vec{0}$

Matrices

Dimensions

$$A^{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \vdots & \ddots & \vdots \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Invertibility

- must be a square matrix to be invertible
- consider matrices A + B
  ↳ if they both have no inverse, AB or BA don't have an inverse either
- $(AB)^{-1} = B^{-1}A^{-1}$, $A^{-1}A = I$ ↳ $[A \mid I]$ ↑ put A in I form
- $(A+B)^{-1} \neq A^{-1} + B^{-1}$

2x2

$$\frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

↑ ad - bc

Column space (Range)

- the columns of matrix $A^{m\times n}$ that span a vector space
- if there is no solution to $A\vec{x} = \vec{b}$, $\vec{b}$ lies outside of columnspace
- usually just the columns of the matrix

* rank = non-zero rows in ref!

Null Space

$N(A) = N(\text{rref}(A))$

- if the nullspace $N(A) = \{\vec{0}\}$, then the matrix has linearly indep. vectors
- else, find the lin. dep. columns!
  ↳ omit these vectors from the column space to find the basis!

the min columnspace

Eigen values

$\det(\lambda I_n - A) = 0$

$\det = ad - bc$

- solve for λ

Eigenvectors

$(A - \lambda I_n)\vec{v} = \vec{0}$

$A\vec{v} = \lambda\vec{v}$

Eigenspace

$E_\lambda = \text{Nullspace}(\lambda I_n - A)$

↳ aka, all the vectors that satisfy the equation $A\vec{v} = \lambda\vec{v}$

* the eigenvalues of an invertible matrix is $\neq 0$

* if an invertible matrix A has eigval λ, A^{-1} has eigval $1/\lambda$!

↳ always true cuz invertible matrix A never $\lambda = 0$

Practice Eig Problem

$A = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$ $\det(\lambda I_n - A) = 0$ ← eigenvalues!

$$\det\left(\begin{bmatrix} \lambda-3 & -2 \\ -4 & \lambda-1 \end{bmatrix}\right) = 0$$

$(\lambda-3)(\lambda-1) - 8 = 0$

$\lambda^2 - 4\lambda - 5 = 0$

$(\lambda-5)(\lambda+1) = 0$

eigenvalues! $\boxed{\lambda = 5, -1}$

$(A - \lambda I_n)\vec{v} = \vec{0}$ ← eigenvectors

$\lambda = 5$:

$$\begin{bmatrix} 5-3 & -2 \\ -4 & 5-1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 2 & -2 \\ -4 & 4 \end{bmatrix}}_{\text{find the null space} = \vec{0}}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\alpha\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for $\alpha \in \mathbb{R}$

$$\begin{bmatrix} 2 & -2 \\ -4 & 4 \end{bmatrix} \xrightarrow{R2+2R1} \begin{bmatrix} 2 & -2 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$v_1 - v_2 = 0$

$v_1 = v_2$

Finding Col + Null Space

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \end{bmatrix}$$

$$C(A) = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}\right)$$

*is this a basis? let's test it!

(if $N(A) = \{0\}$, this is a basis + all cols are lin. independent!

$N(A) = N(\text{rref}(A))$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \end{bmatrix} \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 3 & 2 & 0 \\ 0 & -1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

↑ rref!

$$N(A) = \text{span}\left(\begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}\right)$$

means that cols are NOT lin. indep.

$x_1 = -3x_3 - 2x_4$

$x_2 = 2x_3 + x_4$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3\begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

* Figure out if x_3 + x_4 are lin. indep

$x_3 = 0 \rightarrow x_4 = -1$, $x_1 = 2$, $x_2 = -1$

* x_4 vector is a linear combo!

$$x_1\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = -(-1)\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \checkmark$$

$x_4 = 0 \rightarrow x_3 = -1$, $x_1 = 3$, $x_2 = -2$

[do that again] ✓

* x_3 is lin. combo

∴ basis / min col space is

$$C(A) = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}\right)$$

$$\text{basis} = \left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}\right\}$$