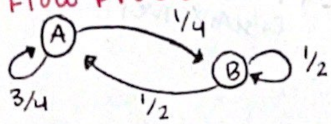
Transition Matrices

Flow problems



$$T = \begin{bmatrix} x_A \\ x_B \end{bmatrix} = \begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix}$$

Things to know

- if all columns add to 1, the system is conservative
- the inverse of T (T^{-1}) does not have the same network w/ flipped arrows
  - ↳ if T has no $\lambda = 0$, T is invertible so matrix C exists where $\vec{s}[n-1] = C\vec{s}[n]$
- Steady state solution
  - ↳ $x_f = \lim_{n \to \infty} T^n \vec{x}[1]$
    - ↳ should satisfy $T\vec{x}_f = \vec{x}_f$
  - ↳ if eigvals < 0, $\lim = 0$
  - ↳ if eigvals $= 0$, lim depends on initial state
  - ↳ if eigvals > 0, $\lim \to \infty$

proofs

Strategy

1. write out any mathematical definitions for what you ~~n~~ know and what you need to show
2. try simple examples to find patterns
3. manipulate the definitions to get from what you know to what to show
4. proof by contradiction
   ↳ assume the opposite of what you need to show, demonstrate that it contradicts what you know!

Example #1

consider square matrix A. prove that if A has a non trivial nullspace, then matrix A is not invertible

$\underbrace{\text{non trivial}}_{\text{the nullspace contains more than just } \vec{0}}$

know: A is square
A has nontrivial null space, aka
$A\vec{x} = \vec{0}$ for some $\vec{x} \neq \vec{0}$

show: A is NOT invertible
↳ $A^{-1}A = I$

prove by contradiction!

assume A is invertible...

$A^{-1}A = I$

$$A\vec{x} = \vec{0}$$
$$A^{-1}A\vec{x} = A^{-1}\vec{0}$$
$$I\vec{x} = A^{-1}\vec{0}$$
$$\vec{x} = \vec{0}$$

shows that if A is invertible, $\vec{x} = \vec{0}$ + $A\vec{x} = \vec{0}$, so by contradiction:
if A is NOT invertible
it can't have $\vec{x} = \vec{0}$ so
A is not invertible!

S is a subspace if:

1. S contains $\vec{0}$ for some $A\vec{x} = \vec{0}$
2. $\vec{x}$ in $S = c\vec{x}$ in S (scalar mult)
3. $\vec{a}$ in S, $\vec{b}$ in S: $\vec{a} + \vec{b}$ in S (vector addition)

Example #2

Given matrix $A^{n \times n}$ with unequal eigenvalues $\lambda_1 \neq \lambda_2$ and corresponding eigenspaces V_1, V_2, the bases of V_1 + V_2 are linearly independent from each other.

know: $\lambda_1 \neq \lambda_2$

show: V_1 + V_2 bases are lin. indep.

prove by contradiction!

assume the bases are linearly dependent on each other, so $\vec{v}_1 = a v_2$, $a \in \mathbb{R}$

*exclude $\vec{0}$ since that is in all vector spaces by def.

$$M\vec{v}_1 = M\vec{v}_1$$
$$M\vec{v}_1 = M(a\vec{v}_2)$$
$$\lambda_1\vec{v}_1 = a(M\vec{v}_2)$$
$$\lambda_1\vec{v}_1 = a(\lambda_2\vec{v}_2)$$
$$\lambda_1(a\vec{v}_2) = a(\lambda_2\vec{v}_2)$$
$$\vec{0} = (\lambda_1 a\vec{v}_2) - (\lambda_2 a v_2)$$
$$\vec{0} = a v_2(\lambda_1 - \lambda_2)$$

↑ must be $\lambda_1 = \lambda_2$ if bases are lin. dep.

∴ proven!

Basis + Linear Independence

$$V = \left\{ \vec{x} \in \mathbb{R}^4 \;\middle|\; \vec{x} = \begin{bmatrix} \alpha \\ \beta \\ \alpha+\beta \\ \alpha-\beta \end{bmatrix}, \text{where } \alpha, \beta \in \mathbb{R} \right\}$$

$$\vec{x} = \alpha\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \beta\begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

dimension = 2 cuz

$$\text{basis} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

(lin indep set $\vec{v}$)

Proof!

know: $a_1v_1 + a_2v_2 \dots + a_nv_n = 0 \to a_1, a_2, a_3, = 0$

prove: $B_1v_1 + B_2(v_2 - v_1) + B_3(v_3 - v_2 - v_1) + \dots = 0$

proof by contradiction

suppose not lin indep
which means $B_1v_1 + B_2(v_2 - v_1) \dots + \dots = 0$
$B \neq \vec{0}$

Matrices

Dimensions

$$A^{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \vdots & & \vdots \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

invertibility

- must be a square matrix to be invertible
- consider matrices A + B
  - ↳ if they both have no inverse, AB or BA don't have an inverse either
- $(AB)^{-1} = B^{-1}A^{-1}$, $\underline{A^{-1}A = I}$ → $[A \mid I]$ ↑ put A in I form
- $(A+B)^{-1} \neq A^{-1} + B^{-1}$

2x2

$$\frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

↑ ad - bc

Column space (Range)

- the columns of matrix $A^{m\times n}$ that span a vector space
- if there is no solution to $A\vec{x} = \vec{b}$, $\vec{b}$ lies outside of columnspace
- usually just the columns of the matrix

* rank = non-zero rows in ref!

Null space

$N(A) = N(\text{rref}(A))$

- if the nullspace $N(A) = \{\vec{0}\}$, then the matrix has linearly indep. vectors
- else, find the lin. dep. columns!
  - ↳ omit these vectors from the column space to find the basis! ↗ the min columnspace

Eigen values

$\det(\lambda I_n - A) = 0$

det = ad - bc

- solve for λ

Eigenvectors

$(A - \lambda I_n)\vec{v} = \vec{0}$

$\underline{\underline{A\vec{v} = \lambda\vec{v}}}$

Eigenspace

$E_\lambda = \text{Nullspace}(\lambda I_n - A)$

↳ aka, all the vectors that satisfy the equation $A\vec{v} = \lambda\vec{v}$

* the eigenvalues of an invertible matrix is $\neq 0$

* if an invertible matrix A has eigval λ, A^{-1} has eigval $1/\lambda$!

↳ always true cuz invertible matrix A never $\lambda = 0$:)

Practice Eig Problem

$A = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$ $\qquad \underline{\det(\lambda I_n - A) = 0}$ ← eigenvalues!

$$\det\left(\begin{bmatrix} \lambda - 3 & -2 \\ -4 & \lambda - 1 \end{bmatrix}\right) = 0$$

$$(\lambda-3)(\lambda-1) - 8 = 0$$

$$\lambda^2 - 4\lambda - 5 = 0$$

$$(\lambda-5)(\lambda+1) = 0$$

eigenvalues! $\boxed{\lambda = 5, -1}$

$\underline{(A - \lambda I_n)\vec{v} = \vec{0}}$ ← eigenvectors

$\underline{\lambda = 5}$:

$$\begin{bmatrix} 5-3 & -2 \\ -4 & 5-1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \alpha\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ for } \alpha \in \mathbb{R}$$

$$\underbrace{\begin{bmatrix} 2 & -2 \\ -4 & 4 \end{bmatrix}}_{\text{find the null space} = \vec{0}}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

:)

$$\begin{bmatrix} 2 & -2 \\ -4 & 4 \end{bmatrix} \xrightarrow{R2+2R1} \begin{bmatrix} 2 & -2 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \qquad \begin{aligned} v_1 - v_2 &= 0 \\ v_1 &= v_2 \end{aligned}$$

Finding Col + Null Space

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \end{bmatrix}$$

$$C(A) = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}\right)$$

* is this a basis? let's test it!

(if $N(A) = \{0\}$, this is a basis + all cols are lin. independent!

$N(A) = N(\text{rref}(A))$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \end{bmatrix} \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 3 & 2 & 0 \\ 0 & -1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

↑ rref!

$$N(A) = \text{span}\left(\begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}\right)$$

means that cols are $\underline{\text{NOT}}$ lin. indep.

$x_1 = -3x_3 - 2x_4$

$x_2 = 2x_3 + x_4$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3\begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

* figure out if x_3 + x_4 are lin indep

$x_3 = 0 \rightarrow x_4 = -1$, $x_1 = 2$, $x_2 = -1$

* x_4 vector is a linear combo! ✓

$$x_1\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = -(-1)\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

$x_4 = 0 \rightarrow x_3 = -1$, $x_1 = 3$, $x_2 = -2$

* x_3 is lin. combo ✓

[do that again]

∴ basis/min col space is

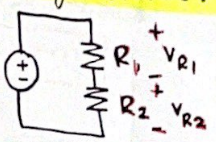
$$C(A) = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}\right)$$

$$\text{basis} = \left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}\right\}$$

Basics: Resistors and Capacitors

	Resistors	Capacitors
General Construction	$R = \rho \frac{\text{length}}{\text{Area}}$	$C = K\varepsilon_0 \frac{\text{Area}}{\text{distance}}$
I-V relation	$V = IR$	$Q = CV \rightarrow I = C\frac{dV}{dt}$
series equivalence	$R_{eq} = R_1 + R_2$	$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$ } $C_1 \parallel C_2 = \frac{C_1 C_2}{C_1 + C_2}$
parallel	opposite →	← opposite
energy stored	resistors don't store energy	$E = \frac{1}{2}CV^2$

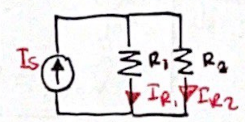
Voltage Divider in series



$V_{R_1} = V_S \frac{R_1}{R_1 + R_2}$

$V_{R_2} = V_S \frac{R_2}{R_1 + R_2}$

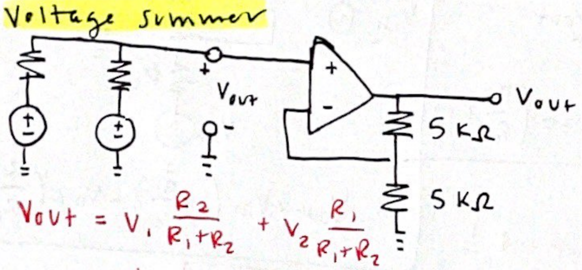
Current Divider in ~~series~~ Parallel



$I_{R_1} = I_S \frac{R_2}{R_1 + R_2}$

$I_{R_2} = I_S \frac{R_1}{R_1 + R_2}$

Voltage Summer



$V_{out} = V_1 \frac{R_2}{R_1 + R_2} + V_2 \frac{R_1}{R_1 + R_2}$

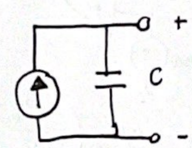
$= \frac{1}{2}V_1 + \frac{1}{2}V_2$ so $V_{out} = V_1 + V_2$

* if $R_1 = R_2$

Capacitors (Basic circuits)

* use $I = C\frac{dV}{dt}$!

$Q = CV$

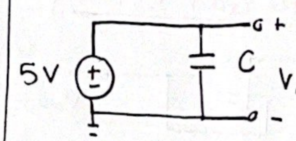


$\frac{d}{dt}Q = \frac{d}{dt}CV$

$I = C\frac{dV}{dt}$

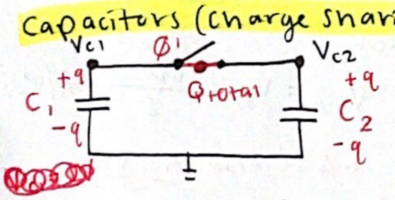
$\therefore V_{out} = \frac{I}{C}t + V_0$

initial charge of capacitor



$Q = CV$!

Capacitors (Charge sharing)



once ϕ_1 closes:

$V_{c1} = 1V = 1V$ } currently
$V_{c2} = 2V$

$C_1 = C_2 = 1\mu F$

$Q_{tot} = +q_{c1} + q_{c2}$

$C_1: q_{c_1}^{\phi_1} = C_1 V_{c1} = (1\mu F)(1V) = 1\mu C$
$C_2: q_{c_2}^{\phi_1} = C_2 V_{c2} = (1\mu F)(2V) = 2\mu C$
} $Q_{tot} = 3\mu C$

$\boxed{V_f = \frac{C_1 V_{c1} + C_2 V_{c2}}{C_1 + C_2}}$ ← general formula!

find V final:

$C_1 V_f + C_2 V_f = 3\mu C$

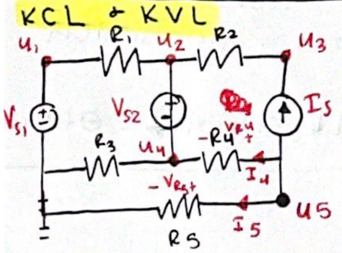
$V_f(C_1 + C_2) = 3\mu C$

$\boxed{V_f = \frac{3}{2}V}$

charge on both caps

$Q = CV \rightarrow Q = 1\mu F(\frac{3}{2}) = \frac{3}{2}\mu C$

KCL + KVL



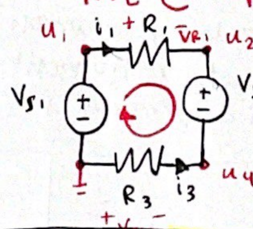
KCL @ u_5

$I_{in} = I_{out}$

$0 = I_S + i_4 + i_5$

$\boxed{0 = I_S + \frac{u_5 - u_4}{R_4} + \frac{u_5}{R_5}}$

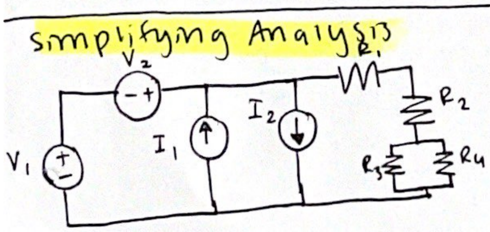
KVL @ upper left



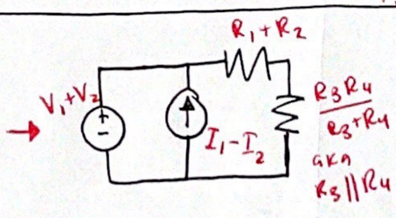
$-V_{S1} + V_{R1} + V_{S2} + V_{R3} = 0$

$\boxed{-V_{S1} + (u_1 - u_2) + V_{S2} + u_4 = 0}$

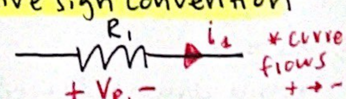
Simplifying Analysis



→



Passive sign convention



* current flows + → -

Power

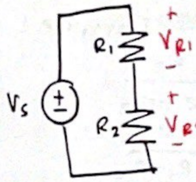
$P = IV$ Always!

* only resistors can use $P = \frac{V^2}{R}$ + $P = I^2R$

* power is always conserved! sum of all power = 0!

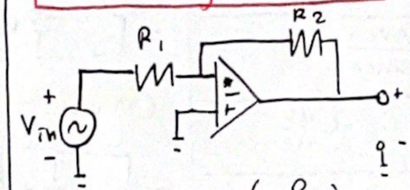
REFERENCE CIRCUITS

Voltage Divider



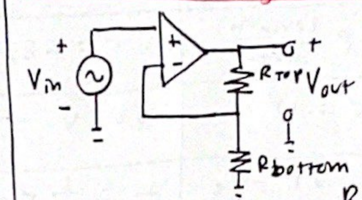
$$V_{R2} = V_S \frac{R_2}{R_1 + R_2}$$

Inverting Op-Amp



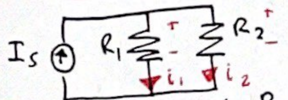
$$V_{out} = V_{in}\left(-\frac{R_2}{R_1}\right)$$

Non-Inverting Op-Amp



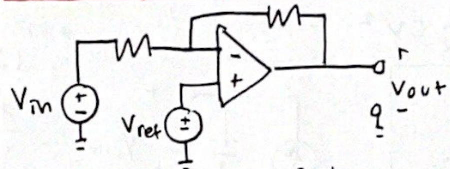
$$V_{out} = V_{in}\left(1 + \frac{R_{top}}{R_{bottom}}\right)$$

Current Divider



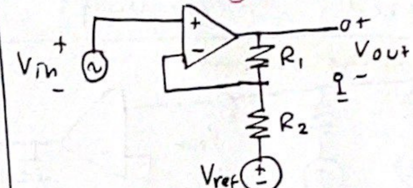
$$I_1 = I_S\left(\frac{R_2}{R_1 + R_2}\right)$$

Inverting Amp w/ Reference



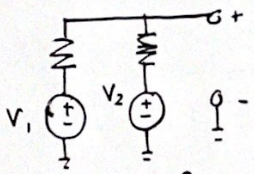
$$V_{out} = V_{in}\left(-\frac{R_f}{R_s}\right) + V_{REF}\left(\frac{R_f}{R_s} + 1\right)$$

Non-Inverting Amp w/ Ref



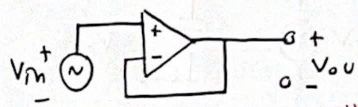
$$V_{out} = V_{in}\left(1 + \frac{R_1}{R_2}\right) - V_{ref}\left(\frac{R_1}{R_2}\right)$$

Voltage Summer



$$V_{out} = V_1\left(\frac{R_2}{R_1 + R_2}\right) + V_2\left(\frac{R_1}{R_1 + R_2}\right)$$

Unity Gain Buffer



Gain = 1 !

$$V_{out} = V_{in}$$

*allows it to give a clearer undistorted signal

Gain

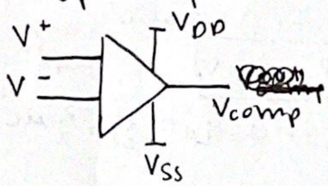
if Gain > 0 → non-inverting

if Gain < 0 → inverting!

Op Amps

→ comparators: compares 2 voltages

→ Op-Amp: operational Amplifier

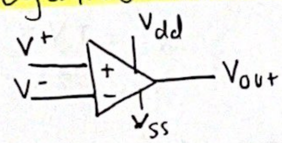


if $V^+ > V_-$; $V_{comp} = V_{DD}$

if $V^+ < V_-$; $V_{comp} = V_{SS}$

- amplifies signals
- isolate circuits to added loading effect
  - ↳ "loading effect" = degree to which the measurement instrument impacts electrical properties of the circuit.

Negative Feedback



$$V_{out} = V_{SS} + \frac{V_{DD} + V_{SS}}{2} + A(V^+ - V^-)$$

* A is ideal! $A = +\infty$

for an op-amp in negative feedback:

$V_{in} - f \cdot V_{out} = V_{err}$ ↑ $= 0$ $\quad \therefore \quad V_{out} = \frac{A}{1 + Af} V_{in}$

checking for negative feedback

① zero out all indep sources
- ↳ voltage source = wire
- ↳ current sources = open switch

② wiggle!
- ↳ if error signal ↓, output ↓ = ⊖ fdbck

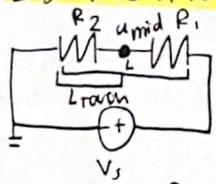
Golden Rules

① $I_+ = I_- = 0$

② $u_+ = u_-$
- ↳ only when in ⊖ feedback
- ↳ $A = +\infty$
- ↳ $V_{error} = 0$ aka $u_+ - u_-$
- ↳ $V_{out} = A\,V_{error} = A(u_+ - u_-)$

$$V_{out} = \frac{A}{1 + Af} u^+ \rightarrow u^- = f V_{out} = \frac{fA}{1 + Af} u^+$$

1D Resistive Touchscreen



$R_2 = \rho \frac{L_{touch}}{A}$

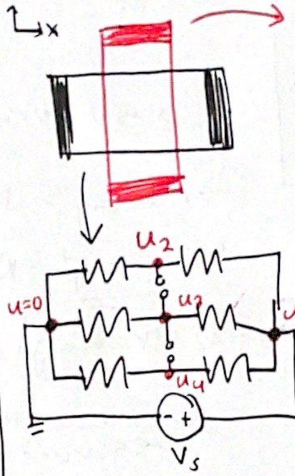
$R_1 = \rho \frac{L_{rest}}{A}$

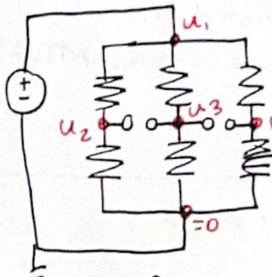
$V_{out} = \frac{R_2}{R_1 + R_2} V_s$ *umid is the touch!

$u_{mid} = \frac{R_2}{R_1 + R_2} V_s = \frac{L_{touch}}{L} V_s$

2D Touch screen (Resistive)

*top = y-coord
bottom = x-coord





$u_3 = \frac{R_{touch}}{R_{rest} + R_{touch}} V_s$

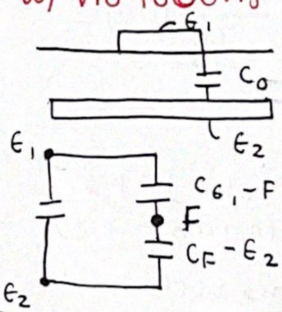
aka $u_3 = \frac{L_{touch}}{L} V_s$ ← vertical for top! ← horizontal for bottom

Superposition

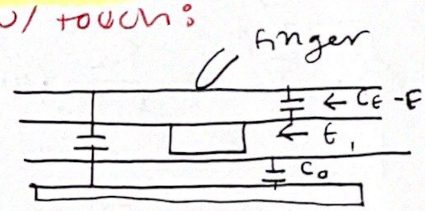
- voltage source = wire
- current source = open switch
- find the value of an element in each "circuit"
- add together the current + voltage

Capacitative Touchscreen

w/ no touch:

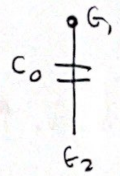


w/ touch:



$(C_{E\parallel} = \| C_{EF - E_2}) + C_0$

w/ no finger



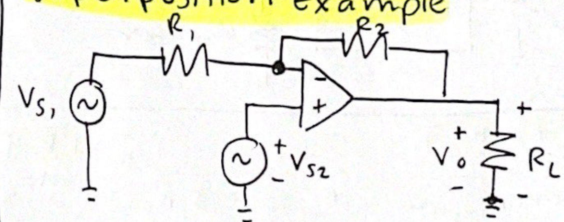
w/ finger equations

$C_0 = \varepsilon \frac{d_2 w_1}{t_1}$

$C_F - E_1 = \varepsilon \frac{d_1 w_1}{t_2 - t_1}$

$C_F - E_2 = \varepsilon \frac{d_2 (w_2 - w_1)}{t_2}$

Superposition Example



$V_o = \underbrace{\left(1 + \frac{R_2}{R_1}\right) V_{s2}}_{V_{s2} \text{ was a non-inverting op amp}} - \underbrace{\left(\frac{R_2}{R_1}\right) V_{s1}}_{V_{s1} \text{ is inverting!}}$

Inner Products

- same as dot product
  - ↳ $\langle \vec{x}, \vec{y} \rangle = \vec{x} \cdot \vec{y} = \vec{x}^T \vec{y}$
  - ↳ $\sum_{i=1}^{n} x_i y_i$, $\langle \vec{x}, \vec{y} \rangle = 0$ only if $\vec{x} = \vec{0}$ cuz $\langle \vec{x}, \vec{x} \rangle \geq 0$

① commutative → $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$

② scalar mult → scale one by a num, scale both, $\langle c\vec{x}, \vec{y} \rangle = c \langle \vec{x}, \vec{y} \rangle$

③ distributive → $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
$\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$

Least Squares

- overdetermined systems (more equations than unknowns)
- decreases noise!
- inconsistent systems:
  - ↳ look for error ($\vec{e} = \vec{b} - A\vec{x}$)

$e \perp x_1 a_1 \rightarrow \langle e, x_1 a_1 \rangle = 0$

$x_1 = \frac{\langle b, a_1 \rangle}{\langle a_1, a_1 \rangle}$ *orthogonal projection!

Distance from Time Delay

$t_i = d_i / v$

$\text{proj}_A(\vec{b}) = A\hat{x} = A(A^T A)^{-1} A^T \vec{b}$

$\text{proj}_a(\vec{b}) = \frac{\vec{b}^T \vec{a}}{\|\vec{a}\|^2} \vec{a}$

(more on back!) ⟶

Random Reminders

- power dissipated from the voltage source is NEGATIVE!

Design Example

Problem

$V_m > 0 \rightarrow$ forward
$V_m < 0 \rightarrow$ backward
$|V_m|$ is proportional to speed

① specs: distance ↓, speed ↓

we want decreasing positive voltage for this

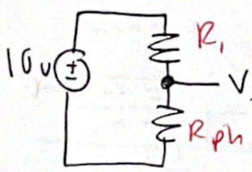
$V_m \geq 5V$ when "far away"

"far away" $R_{photosensor} = 10\,k\Omega$
"nearby" $R_{photosensor} = 100\,\Omega$

② strategy as $D\uparrow$, $L\downarrow$, $R_{photosensor}\uparrow$

we need to build something that measures resistance → output voltage!
↳ try voltage divider!

③ solve



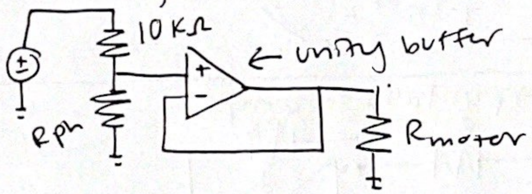
$$V_x = \frac{R_2}{R_1+R_2}(10V) = \frac{1}{1+R_1/R_2}(10V)$$

as $R_1\uparrow$, $\left(\frac{R_1}{R_2}\right)\uparrow$, $V_x\downarrow$

$R_2\uparrow$, $\left(\frac{R_1}{R_2}\right)\downarrow$, $\underbrace{V_x\uparrow}_{\text{make } R_2 = R_{ph}}$

* need $V_x = 5V$ when $R_{ph} = 10\,k\Omega$

use a unity buffer! to isolate it :)



$D\uparrow$, $L\downarrow$, $R_{ph}\uparrow$, $\left(\frac{R_1}{R_{ph}}\right)\downarrow$, $V_x\uparrow$ ✓

Least squares (cont.)

- e has to be orthogonal to space x, a,
- a vector is orthogonal to every vector in the column space of A if/only if it is orthogonal to each of the columns $\vec{a}_i$ that for basis of column space

$A^T\vec{e} = \vec{0}$

$A^T(\vec{b} - A\vec{x}) = \vec{0}$

$A^TA\vec{x} = A^T\vec{b} \rightarrow \vec{x} = (A^TA)^{-1}A^T\vec{b}$

$\text{Null}(A^TA) = \text{Null}(A)$

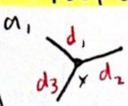
$A^TA\vec{v} = \vec{0}$

$\vec{v}^TA^TA\vec{v} = \vec{v}^T\vec{0} = 0$

$(A\vec{v})^T(A\vec{v}) = 0 \rightarrow \langle A\vec{v}, A\vec{v}\rangle$

$= \|A\vec{v}\|^2 = 0$

Trilateration



$\|\vec{x} - \vec{a}_1\|^2 = d_1^2$
$\|\vec{x} - \vec{a}_2\|^2 = d_2^2$
$\|\vec{x} - \vec{a}_3\|^2 = d_3^2$

① $x^Tx - 2a_1^Tx + \|a_1\|^2 = d_1^2$
② subtract from eachother to get!

*beacons → $(x-a)^2 + (y-b)^2 = r^2$

Cauchy-Schwarz Inequality

$\langle\vec{x}, \vec{y}\rangle \leq \|\vec{x}\|\|\vec{y}\|$

$|\langle\vec{x}, \vec{y}\rangle| = |\,\|\vec{x}\|\|\vec{y}\|\cos\theta\,|$
$= \|\vec{x}\|\|\vec{y}\||\cos\theta|$
$\leq \|\vec{x}\|\|\vec{y}\|$

Cross correlation

$\text{corr}_x(\vec{y})[k]$
$$= \sum_{i=-\infty}^{\infty} x[i]\,y[i-k]$$

- small subscript stays the same
- large one moves